STATISTICAL PHYSICS OF GEOMETRICALLY FRUSTRATED MAGNETS

Classical spin liquids, emergent gauge fields and fractionalised excitations

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Outline

- Geometrically frustrated magnets

  Experimental signatures of frustration

- Classical models

  Degeneracy of under-constrained ground states

  Ground state selection: order from disorder

- **Low temperature correlations**

  Mean field theory & large-n theory

  Emergent fields & fractionalised excitations

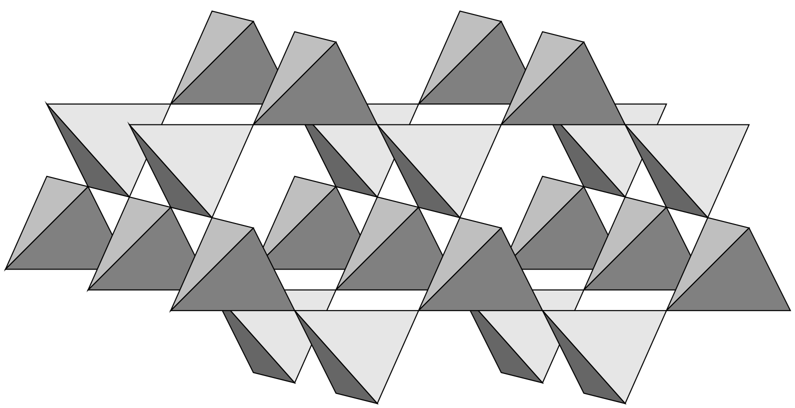
    In 2D — for triangular lattice Ising antiferromagnet

    In 3D — for spin ice

Correlations induced by ground state constraints

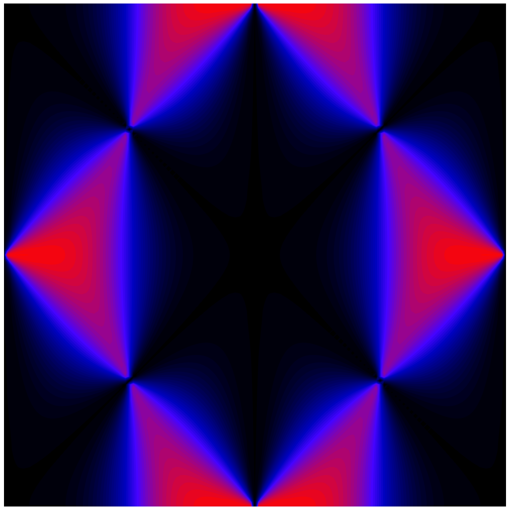
Local constraints

$$\sum_{tet} \mathbf{S}_i = 0$$



Long range correlations

Sharp structure in $\langle \mathbf{S}_{-\mathbf{q}} \cdot \mathbf{S}_{\mathbf{q}} \rangle$



Theoretical description of low-T state

Mean field theory?

Recall mean field approach:

Replace full Hamiltonian $\mathcal{H}$ by single-spin approximation $\mathcal{H}_0$

$$\mathcal{Z}^{-1}\mathrm{Tr}\left(e^{-\beta\mathcal{H}}\ldots\right) \equiv \langle\ldots\rangle \;\Rightarrow\; \mathcal{Z}_0^{-1}\mathrm{Tr}\left(e^{-\beta\mathcal{H}_0}\ldots\right) \equiv \langle\ldots\rangle_0$$

Variational free energy

$$F \leq \langle\mathcal{H}\rangle_0 - TS_0 = \sum_{ij} J_{ij} m_i m_j + c k_\mathrm{B} T \sum_i m_i^2 + \ldots$$

$$\equiv \underline{m}^\mathrm{T} \cdot (\mathbb{J} + c k_\mathrm{B} T\, \mathbb{I}) \cdot \underline{m} + \ldots$$

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Mean field theory?

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Variational free energy

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Pick $\{m_i\}$ to minimise estimate for F

High T: $m_i = 0$ **Low T:** $m_i \neq 0$

Spectrum of $\mathbb{J}$ fixes mean field T_c and ordering pattern

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Spectrum of $\mathbb{J}$ **fixes mean field** T_c **and ordering pattern**

Geometric frustration

$\Rightarrow$ **flat lowest band in** $\mathbb{J} \Rightarrow$ **ordering undetermined**

Theoretical description of low-T state

Self-consistent Gaussian approximation (large-n limit)

Soften constraint on spin lengths:

$$\mathrm{Tr}\ldots \quad \equiv \prod_i \int \mathrm{d}\,\vec{S}_i\,\delta(|\vec{S}_i|-1)\ldots \approx \prod_i \int \mathrm{d}\,\vec{S}_i\,e^{-\frac{\lambda}{2}|\vec{S}_i|^2}\ldots$$

— with λ chosen so that $\langle|\vec{S}_i|^2\rangle = 1$

Then

$$\langle\ldots\rangle = \mathcal{Z}^{-1}\int \mathrm{d}\,\{S_i\}\ldots e^{-\frac{1}{2}S^{\mathrm{T}}(\beta\mathbb{J}+\lambda\mathbb{I})S}$$

Theoretical description of low-T state

Self-consistent Gaussian approximation (large-n limit)

$$\langle \ldots \rangle = \mathcal{Z}^{-1} \int \mathrm{d}\{S_i\} \ldots e^{-\frac{1}{2} S^{\mathrm{T}} (\beta \mathbb{J} + \lambda \mathbb{I}) S}$$

— with λ chosen so that $\langle |\vec{S}_i|^2 \rangle = 1$

So that

$$\langle S_i S_j \rangle = \left[(\beta \mathbb{J} + \lambda \mathbb{I})^{-1} \right]_{ij}$$

with λ fixed by

$$1 = N^{-1} \mathrm{tr} (\beta \mathbb{J} + \lambda \mathbb{I})^{-1}$$

Theoretical description of low-T state

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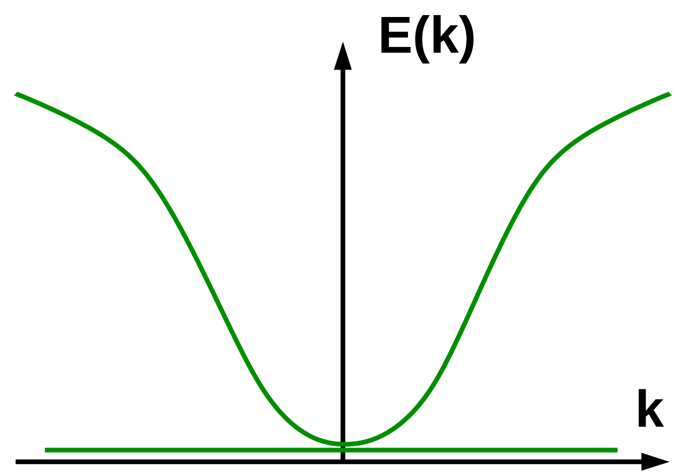
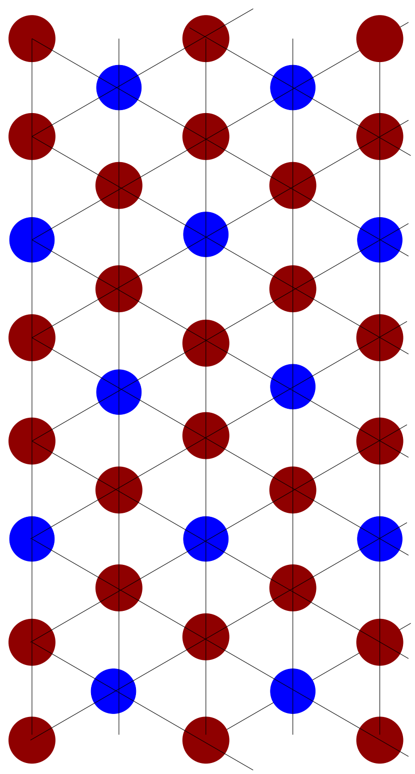


Low T: correlator is projector $\mathbb{P}$ onto flat band $\langle S_i S_j \rangle \propto \mathbb{P}_{ij}$

Ground states of TLIAFM

Triangular lattice Ising antiferromagnet is disordered at $T = 0$

Six $\sqrt{3} \times \sqrt{3}$ ordered states



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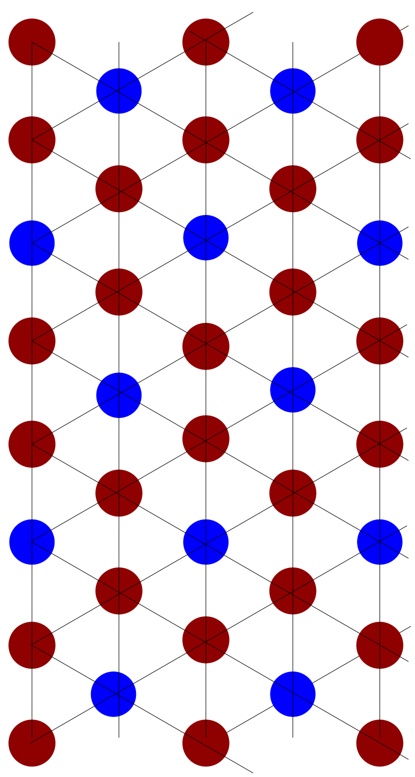
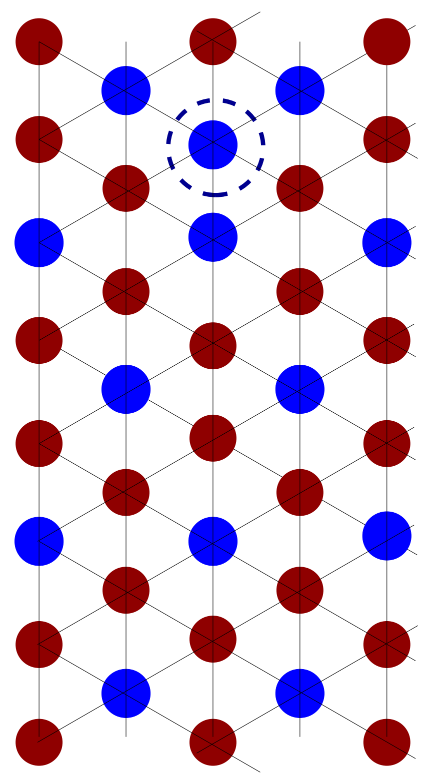


with defects at no energy cost



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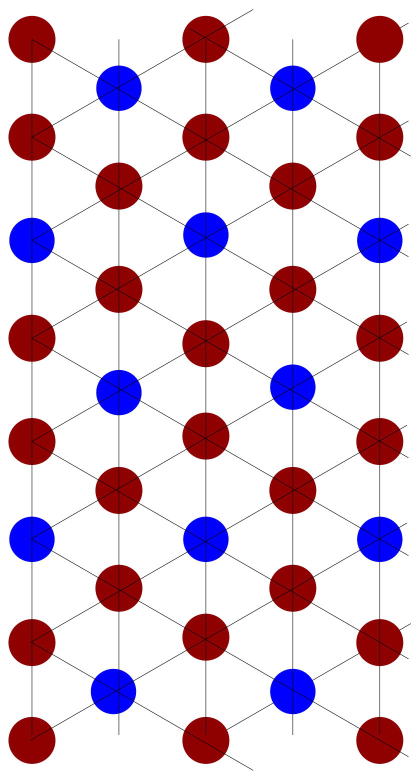
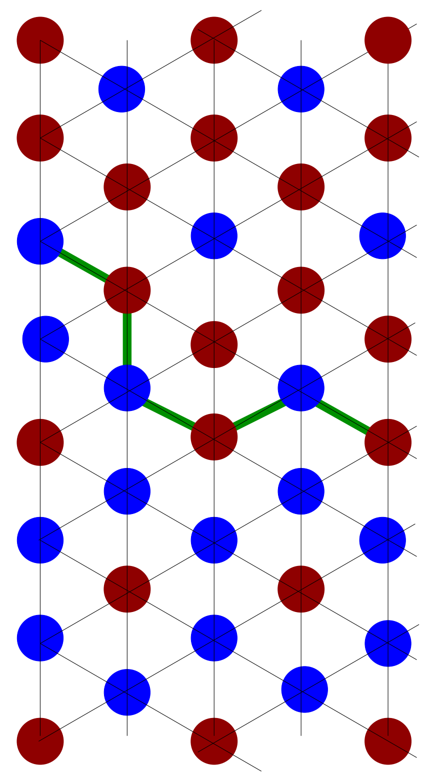


and domain walls at no energy cost



. . . but with entropy cost

TLIAFM & height model

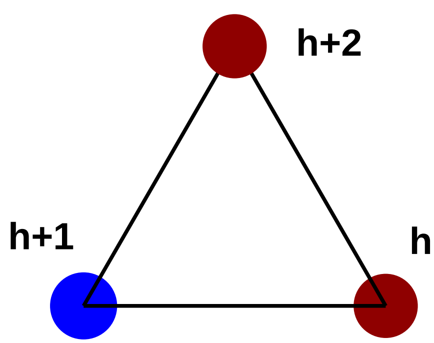
Blöte and Hilhorst (1982)

Height differences

e.g. clockwise around up triangle

$\Delta h = -2$ **parallel spins**

$\Delta h = +1$ **opposite spins**



Heights at triangle centres

$h(\mathbf{r}) =$ **integer mod 6**

$h(\mathbf{r})$ **is flat in the six** $\sqrt{3} \times \sqrt{3}$ **states**

– has steps of ± 1 **at domain walls**

Ground state fluctuations: entropic weight

$$P[h(\mathbf{r})] \sim e^{-\mathcal{H}} \quad \textbf{with} \quad \mathcal{H} = \frac{K}{2} \int \mathrm{d}^2\mathbf{r}\, |\nabla h(\mathbf{r})|^2$$

TLIAFM & height model

Spins in terms of heights

$$\sigma_{\mathbf{r}} \sim \cos[\pi h(\mathbf{r})/3 + \varphi_{\mathbf{r}}]$$

Spin correlations

$$\langle \sigma_{\mathbf{r}} \sigma_{\mathbf{r}'} \rangle \sim |\mathbf{r} - \mathbf{r}'|^{-1/2} \times \begin{cases} +1 & \text{same sublattice} \\ -1/2 & \text{different sublattices} \end{cases}$$

Discreteness of heights $\Rightarrow$ pinning potential

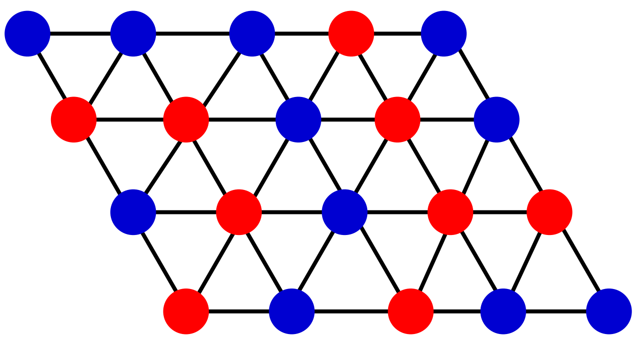
$$\mathcal{H} \Rightarrow \mathcal{H} - g \int \mathrm{d}^2\mathbf{r} \, \cos 2\pi h(\mathbf{r})$$

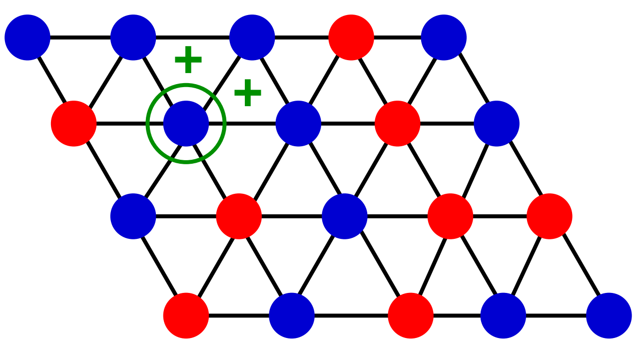
— irrelevant under RG

Excitations in TLIAFM & height model

triangles with three spins parallel $\equiv$ height field vortices

One spin flip creates vortex-antivortex pair





Height differences

clockwise around up triangle

$\Delta h = -2$ **parallel spins**

$\Delta h = +1$ **opposite spins**

Height changes by ± 6

around down/up triangle

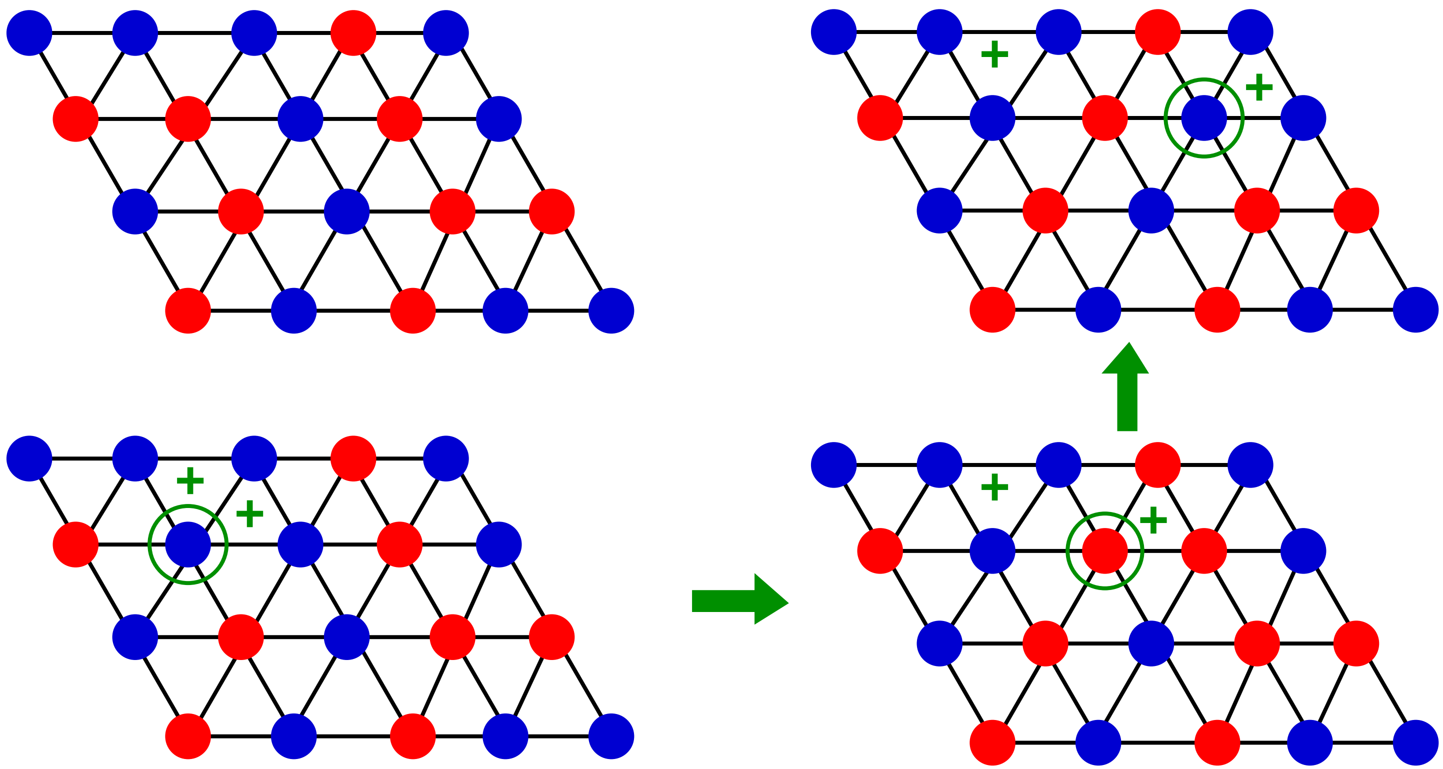
with all spins parallel

Excitations in TLIAFM & height model

triangles with three spins parallel $\equiv$ height field vortices

One spin flip creates vortex-antivortex pair

Further spin flips separate vortex-antivortex pair



Interaction between vortex-antivortex pairs

$$P[h(\mathbf{r})] \sim e^{-\mathcal{H}} \quad \textbf{with} \quad \mathcal{H} = \frac{K}{2} \int \mathrm{d}^2\mathbf{r}\, |\nabla h(\mathbf{r})|^2$$

For isolated vortex at origin

$$|\nabla h(\mathbf{r})| = \frac{6}{2\pi r}$$

In system of size L

$$\int \mathrm{d}^2\mathbf{r}\, |\nabla h(\mathbf{r})|^2 \propto \log(L)$$

Log interaction potential between vortices

$$V(R) \propto K \log R$$

— but also entropy gain

$$2 \log R$$

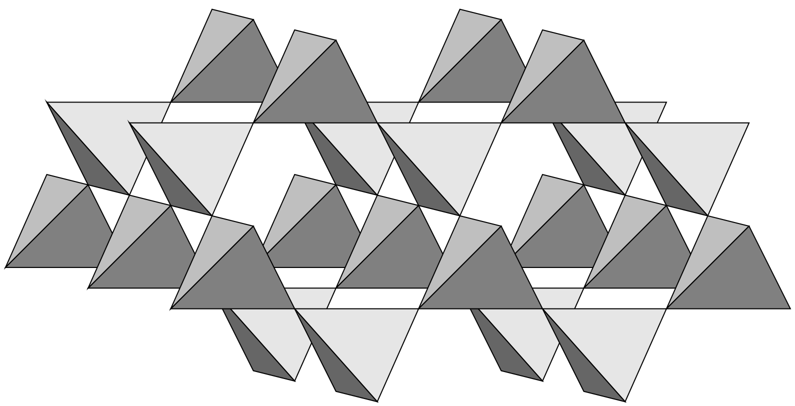
from separation

Unbound for small K $\Rightarrow$ **Correlation length** $\xi \sim \exp(4\beta J)$

Correlations and excitations in 3D

Local constraints

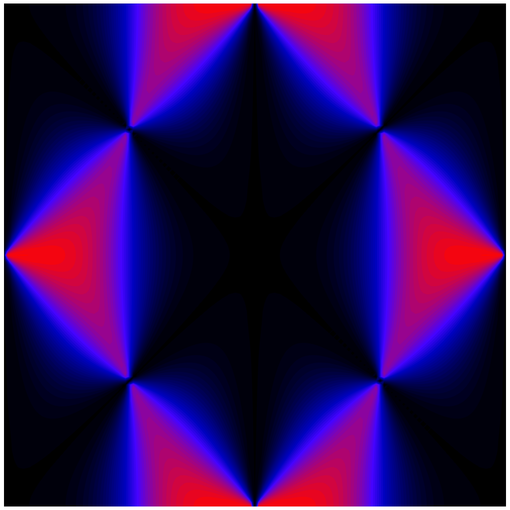
$$\sum_{tet} \mathbf{S}_i = 0$$



Long range correlations

Sharp structure in

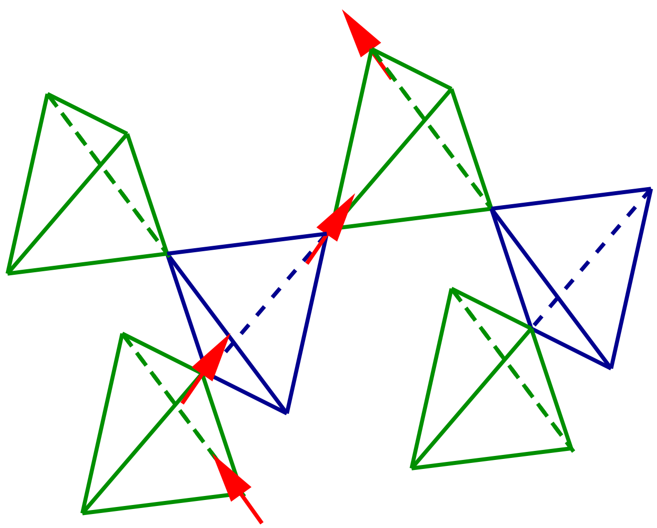
$$\langle \mathbf{S}_{-\mathbf{q}} \cdot \mathbf{S}_{\mathbf{q}} \rangle$$

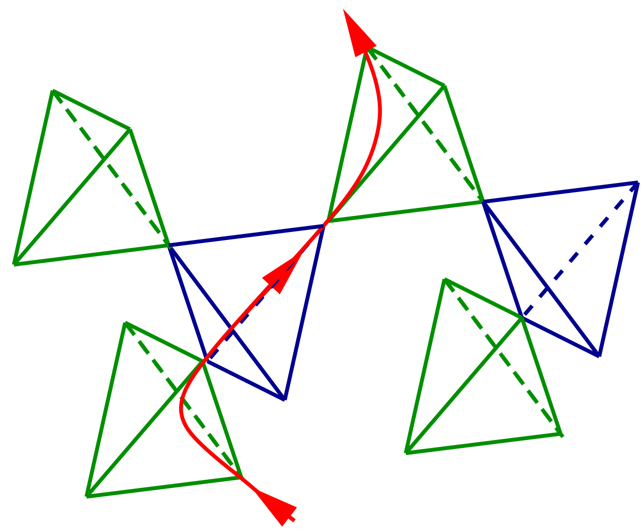


Gauge theory of ground state correlations

Youngblood *et al* (1980), Huse *et al* (2003), Henley (2004)

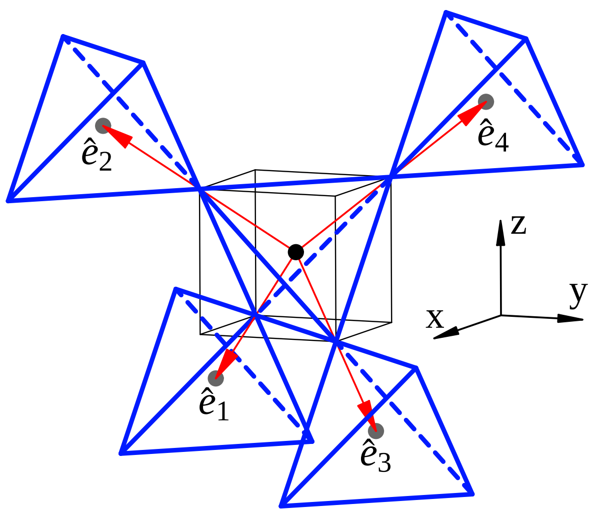
Map spin configurations . . . **. . . to vector fields** $\mathbf{B}(\mathbf{r})$





'two-in two out' groundstates . . . **. . . map to divergenceless** $\mathbf{B}(\mathbf{r})$

Details of mapping



Construct vector fields $\vec{B}^l$ from each spin component S^l :

$$\vec{B}^l_i = \hat{e}_i S^l_i$$

Ground state constraint becomes flux conservation law:

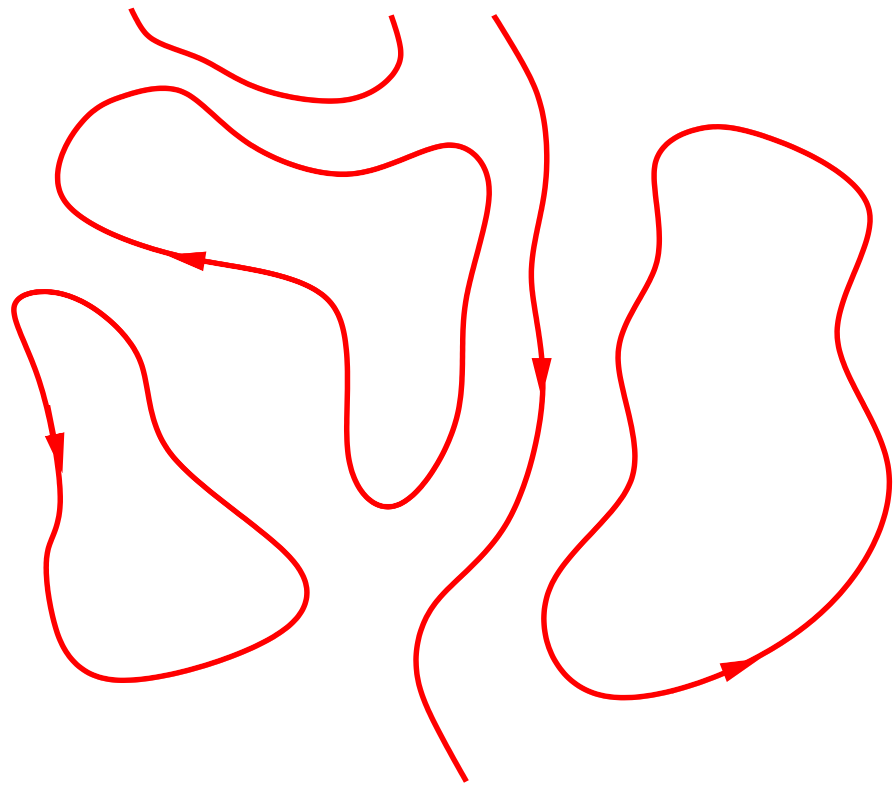
$$\sum_{tet} S^l_i = 0 \rightarrow \nabla \cdot \vec{B}^l = 0$$

$$\vec{B}^l = \nabla \times \vec{A}^l$$

Coarse-grained distribution:

$$P(\vec{A}) \propto \exp(-\tfrac{\kappa}{2} \int [\nabla \times \vec{A}]^2)$$

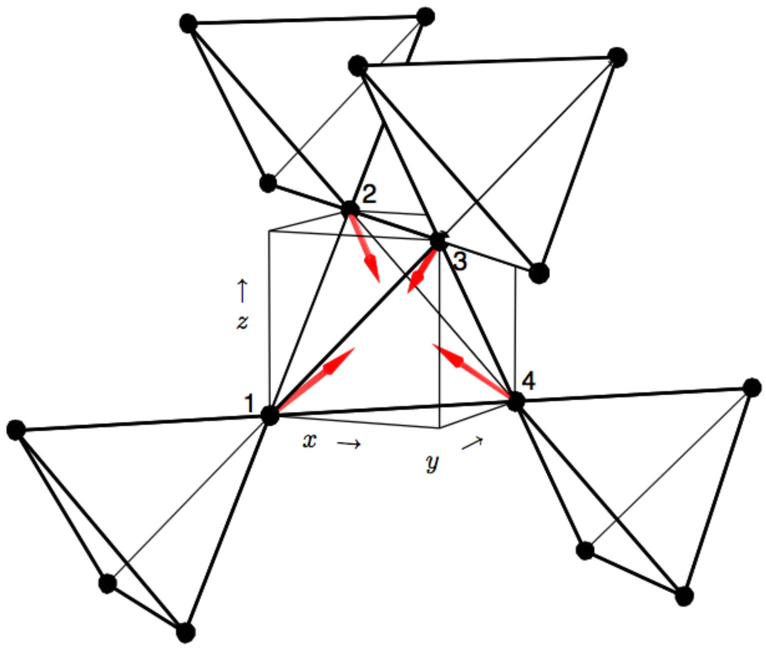
Ground states as flux loops



Entropic distribution: $P[\mathbf{B}(\mathbf{r})] \propto \exp(-\frac{\kappa}{2} \int \mathbf{B}^2(\mathbf{r}) \mathrm{d}^3\mathbf{r})$

Power-law correlations: $\langle B_i(\mathbf{r}) B_j(\mathbf{0}) \rangle = \frac{3 r_i r_j - r^2 \delta ij}{4\pi\kappa\, r^5}$

Translating between fluxes and spins



$$\begin{pmatrix} M(\mathbf{q}) \\ B_x(\mathbf{q}) \\ B_y(\mathbf{q}) \\ B_z(\mathbf{q}) \end{pmatrix} = \mathsf{R} \cdot \begin{pmatrix} S_1(\mathbf{q}) \\ S_2(\mathbf{q}) \\ S_3(\mathbf{q}) \\ S_4(\mathbf{q}) \end{pmatrix}$$

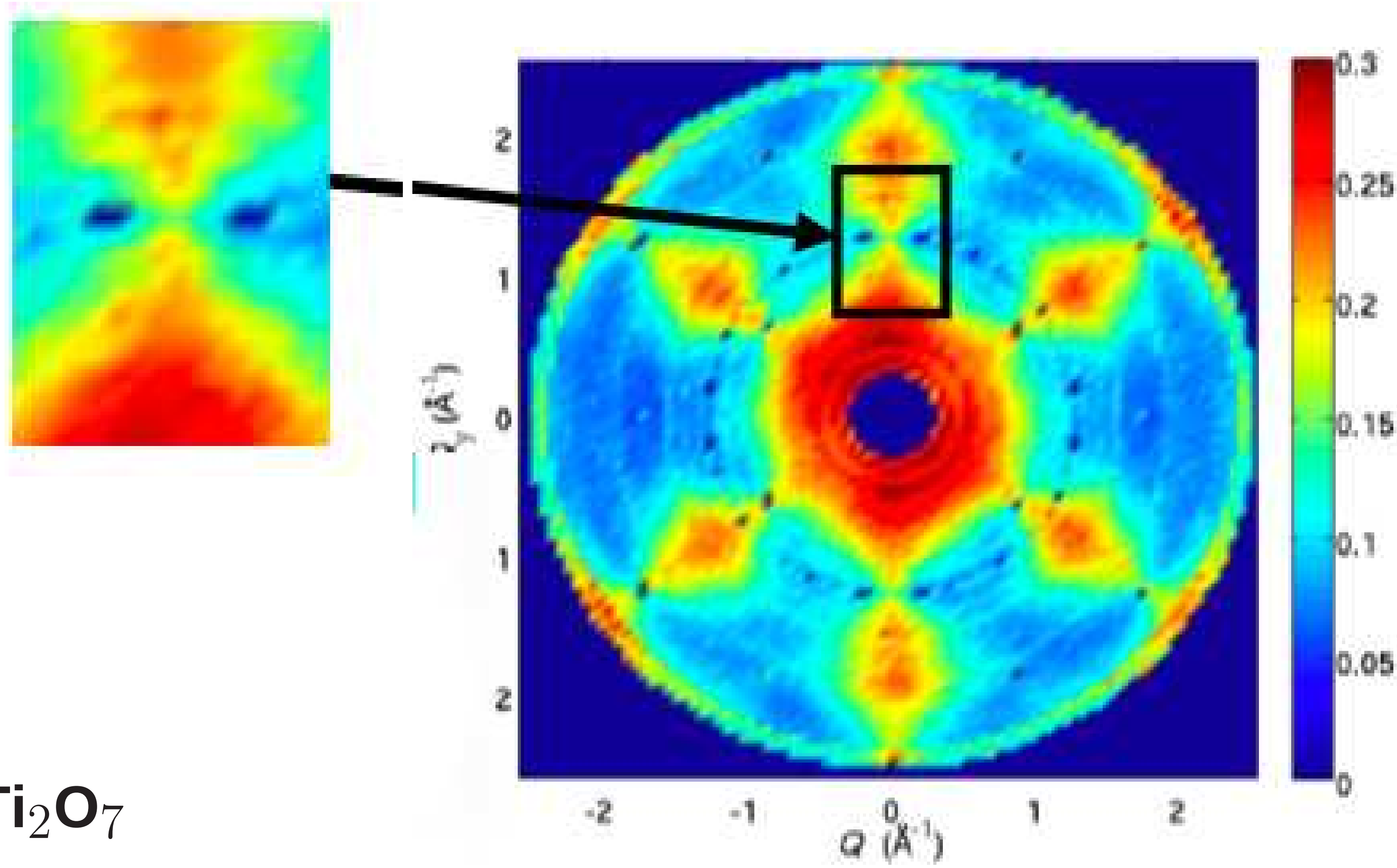
$$\mathsf{R} = \mathsf{R}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$S_j(\mathbf{K} + \mathbf{q}) = e^{i\mathbf{K}\cdot\mathbf{r}_j} S(\mathbf{q})$$

Small-$\mathbf{q}$ structure in $\vec{B}(\mathbf{q})$ appears near Bragg points $\mathbf{K}$ with $\mathbf{K} \neq 0$

Low T correlations from neutron diffraction

Fennell, Bramwell and collaborators (2009)

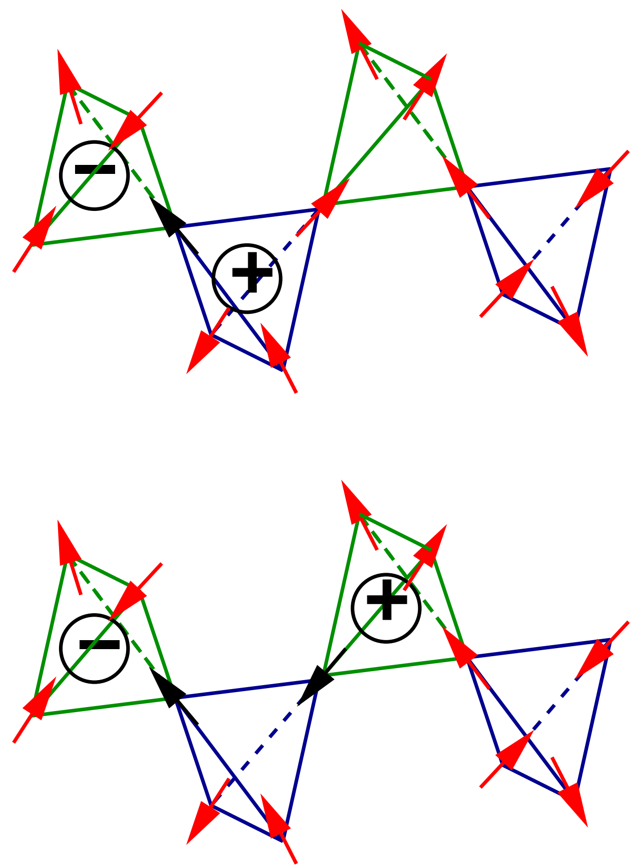


$Ho_2Ti_2O_7$

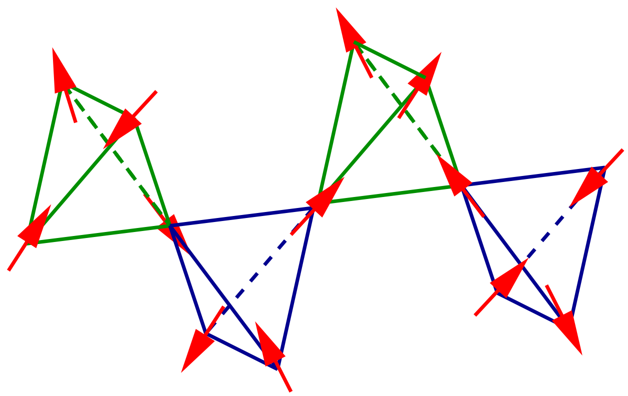
Monopoles in spin ice

Monopole excitations

Excited states



Ground state



Castelnovo, Moessner and Sondhi (2008)

Interactions between monopoles

Interactions from two origins:

- **Influence of monopoles on entropy of spin ice ground states**

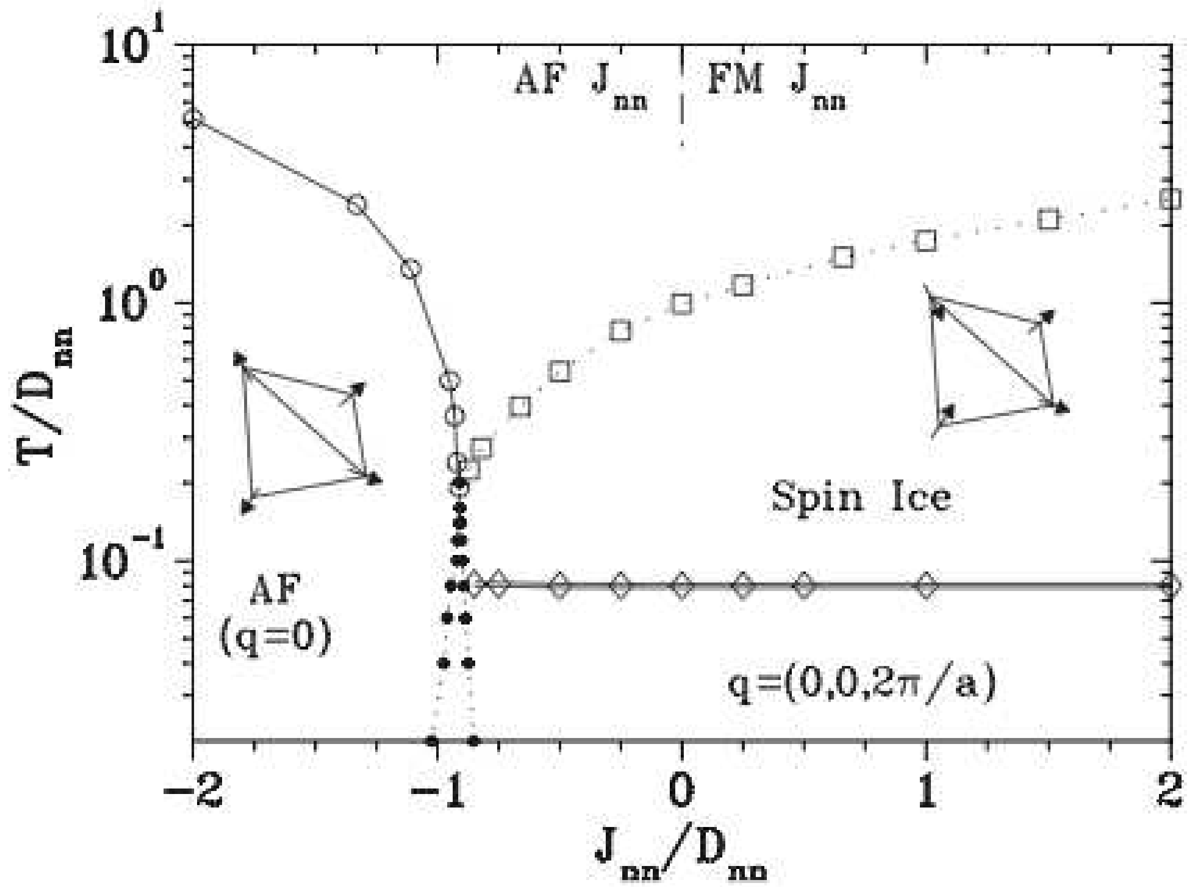
$$P[\mathbf{B}(\mathbf{r})] \propto \exp(-\tfrac{\kappa}{2}\int \mathbf{B}^2(\mathbf{r})\mathrm{d}^3\mathbf{r})$$

— **implies** $\beta V(R) \propto R^{-1}$

- **Effects of further neighbour (dipolar) spin interactions**

— **lifts ground state degeneracy of nearest-neighbour model**

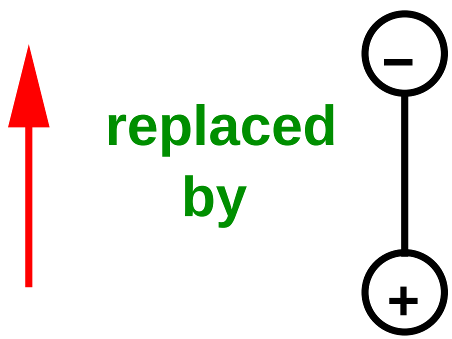
Effect of dipolar interactions on equilibrium behaviour in spin ice

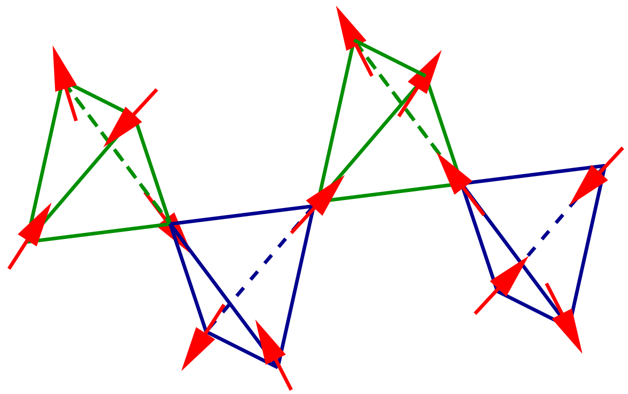


Melko and Gingras (2004).

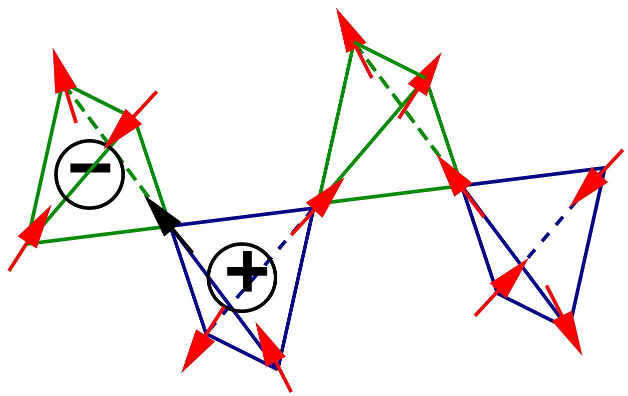
Coulomb potential between monopoles from dipolar spin interactions

View spins as extended dipoles





zero net charge

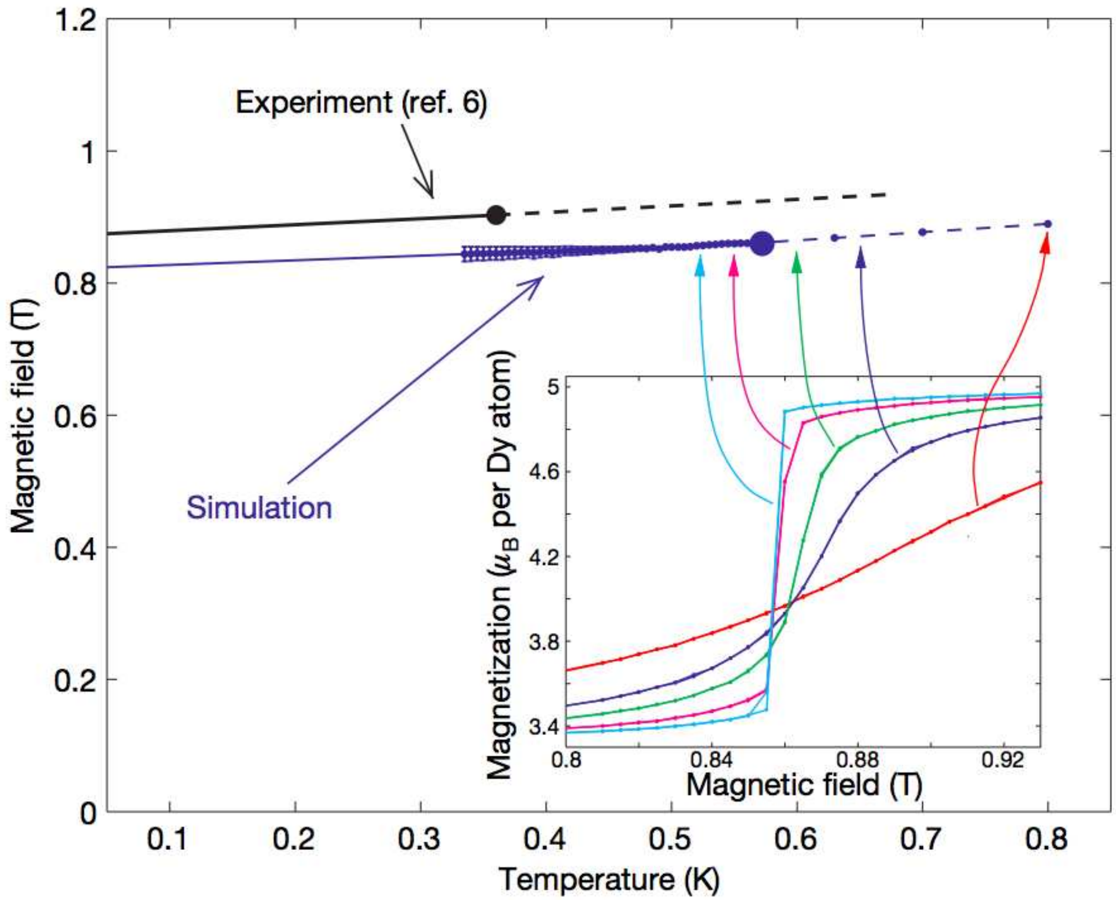


non-zero net charges

Probing interactions between monopoles

Use $[111]$ magnetic field to control monopole density

— observe monopole 'liquid-gas' transition



Castelnovo, Moessner and Sondhi, Nature 451, 42 (2008).

Summary

Geometric frustration

leads to macroscopic classical ground state degeneracy

possibility of order-by-disorder . . . but long-range order avoided

At low T: strong correlations + large fluctuations

emergent degrees of freedom within ground-state manifold

stable power-law correlations

fractionalised excitations

Coulomb interactions from dipolar coupling