Many-body quantum dynamics: of entanglement, of operator spreading, of thermalization.

Strongly-interacting and highly-excited systems:

- condensed matter
- many-body AMO systems
- spin chains are "simplest" examples.

Generic, local non-integrable, chaotic systems.
not near ground states.

In regimes where "quasiparticle" description is not appropriate.

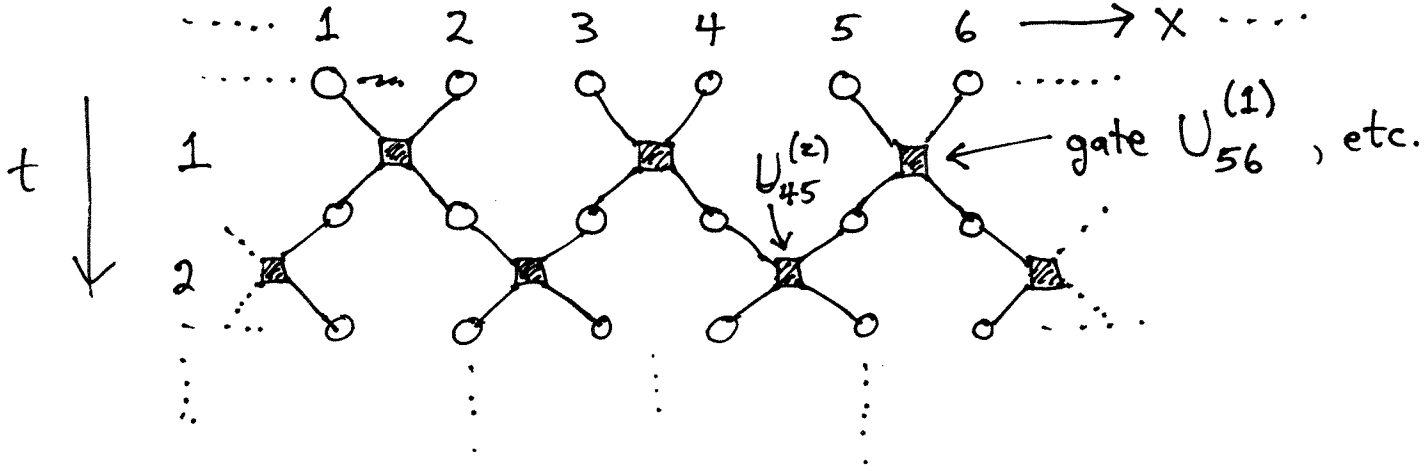
Unitary dynamics ("closed" systems): $\rho(0)$ = initial state

$$\rho(t) = U(0,t)\,\rho(0)\,U^{\dagger}(0,t)$$

Cases:

- time-independent Hamiltonian: $U(0,t) = e^{-iHt}$, conserved energy.
- Floquet systems: $H(t) = H(t+T)$ $\quad U(0,nT) = [U(0,T)]^n$ $\quad T$ = period, n = integer

- nonperiodic $H(t)$: e.g. random quantum circuits:



← "qudits" with q states per site: $q \to \infty$ allows analytical treatment (e.g. Nahum et al. 2017)

What kinds of questions do we want to address?:

Start with initial state $|\psi(0)\rangle$ of <u>low</u> entanglement and/or <u>low</u> entropy (non-equilibrium). (<u>low</u> relative to thermal equilibrium)

Does the system act as a "bath" for itself and bring itself to thermal equilibrium with $U(0,t)$ for $|t| \to \infty$?

If so, what are the dynamics of its entanglement?

How does hydrodynamics "emerge" from the unitary reversible microscopic dynamics?

Or: Start with an initially local operator:

$$A_0(0) = \cdots \otimes \mathbb{1}_{-2} \otimes \mathbb{1}_{-1} \otimes A_0 \otimes \mathbb{1}_1 \otimes \mathbb{1}_2 \otimes \cdots$$

($A_0(0)$: $x = 0$, $t = 0$; sites: $\cdots -2, -1, 0, 1, 2 \cdots$)

What does this operator become under time evolution?

(How does system "hide" $\langle A_0 \rangle_{t=0}$?)

$$A_0(t) = U^\dagger(0,t)\, A_0(0)\, U(0,t)$$

$$= \text{what, when written in terms of the local operators?}$$

Operator "spreads" inside a "Lieb-Robinson cone" and becomes highly entangled:

$$A_0(t) \simeq \cdots \otimes \mathbb{1}_{x=-v_{LR}|t|} \otimes \left[\text{operator acting on } -v_{LR}|t| < x < v_{LR}|t|\right] \otimes \mathbb{1}_{x=v_{LR}|t|} \otimes \mathbb{1}_{x=v_{LR}|t|+1} \otimes \cdots$$

Thermalization.

Thermal equilibrium: maximum entropy given a few constraints,

such as: energy conservation $\longleftrightarrow$ temperature

particle number $\longleftrightarrow$ chemical potential

$\vdots$

Many systems do thermalize: $U(0,t)$ for $|t| \to \infty$ (closed) does bring the system to thermal equilibrium. (in limit of large systems)

Systems that fail to thermalize include:

integrable systems: ∞ number of conservation laws

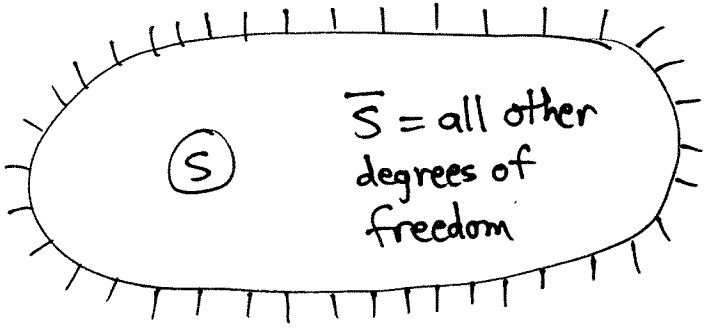
one case is many-body localization (MBL): tomorrow.

Also: "prethermalization": a very long transient regime before thermalization.

Today: restrict our attention to systems that thermalize well:

[redacted] "quantum chaotic" systems; act as "thermal bath" for themselves.

Thermalization: "need" $\rho(0)$ with sub-extensive uncertainty in any conserved quantity. Look at small subsystem S



$$\rho_S(t) \equiv \operatorname{Trace}_{\bar{S}} \rho(t) \overset{\text{in long/time/large volume } \bar{S} \text{ limits}}{=} \rho_S^{(eq)}(T, \{\mu_i\})$$

$$\equiv \operatorname{Trace}_{\bar{S}} \frac{e^{-\beta(H - \sum_i \mu_i N_i)}}{Z}$$

or whichever thermal equilibrium ensemble you prefer

subsystems go to thermal equilibrium, even though the full system "remembers" every detail of the initial state.

If system is Hamiltonian or Floquet and all initial states do thermalize, then eigenstates of $U(0,T)$ must be thermal:

Eigenstate Thermalization Hypothesis (E.T.H.) Jensen + Shankar '85, Srednicki '94

All eigenstates of U are thermal. (many-body eigenstates)

For thermalizing systems when ETH is true:

Each individual many-body eigenstate of $U(0,T)$ is a fine equilibrium microcanonical ensemble: "single-eigenstate ensembles"

If all eigenstates of U are thermal, how do we make non-thermal initial state?

$$U(0,t)\,|n\rangle = e^{i\phi_n(t)}\,|n\rangle \qquad \text{each } |n\rangle \text{ is thermal}$$

$$\rho(0) = \sum_{n,m} |n\rangle\, \rho_{nm} \,\langle m|$$

↑ to have non-thermal $\rho(0)$ the phases of these coherences ($n \neq m$) need to be "finely tuned" to a low-entropy state.

Then:

$$\rho(t) = \sum_{n,m} |n\rangle\, \rho_{nm}\, \underbrace{e^{i(\phi_n(t)-\phi_m(t))}}\, \langle m|$$

↑ At long times these phases become essentially random:

"Equilibration is de-phasing" $\rho(t)$ becomes thermal.

Dynamics of thermalization:

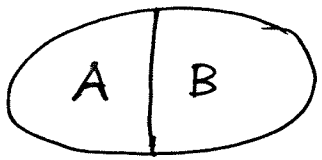
1) Transport of conserved quantities (old and well-formulated)

→ 2) Dynamics of entanglement ← (.... Nahum, et al. 2017) PRX 7, 031016

Generic local interactions increase entanglement until it is thermalized (~~...~~ entropy is maximized).

Quantifying entanglement of a pure state $|\Psi\rangle$:

Choose a "local" cut:



$\rho_A = \underset{B}{\text{Trace}} |\Psi\rangle\langle\Psi|$

Entanglement is a property of the state and the cut, given by ρ_A and ρ_B

"Entanglement spectrum" (Li + Haldane '08) is spectrum of ρ_A, ρ_B

also Schmidt eigenstates of ρ_A, ρ_B

Convenient measures of entanglement entropy S:

von Neumann: $S = -\text{Trace}\{\rho_A \log \rho_A\}$

2nd Renyi: $S_2 = -\log \text{Trace}\, \rho_A^2$

Consider:

Finite 1D system in state $|\psi(t)\rangle$, cut at x:

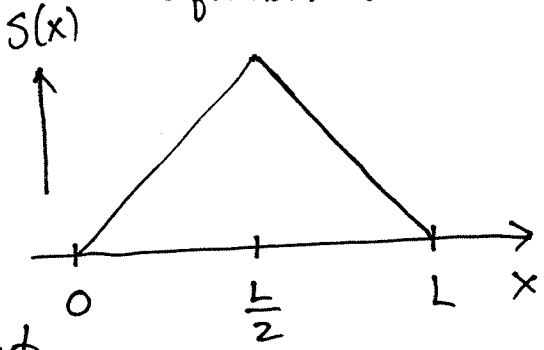
entanglement entropy $S(x,t)$

To look only at entanglement dynamics, assume conserved densities (if any) are at thermal equilibrium, but entanglement is initially sub-thermal. Call $|t| = t \geq 0$.

At thermal equilibrium, entropy density is s_{eq}

$$\left|\frac{dS(x)}{dx}\right| = s_{eq}$$

equilibrium:



Where $\left|\frac{\partial S(x,t)}{\partial x}\right| < s_{eq}$, dynamics produces entanglement.

→ Coarse-grained entanglement "hydrodynamics":

$$\frac{\partial S(x,t)}{\partial t} = s_{eq}\left[\Gamma\left(\frac{\partial S(x,t)}{\partial x}\right) + \cdots \left(\frac{\partial^2 S}{\partial x^2}\right)\cdots\right]$$

this function gives local entanglement ~~rate~~ production rate.

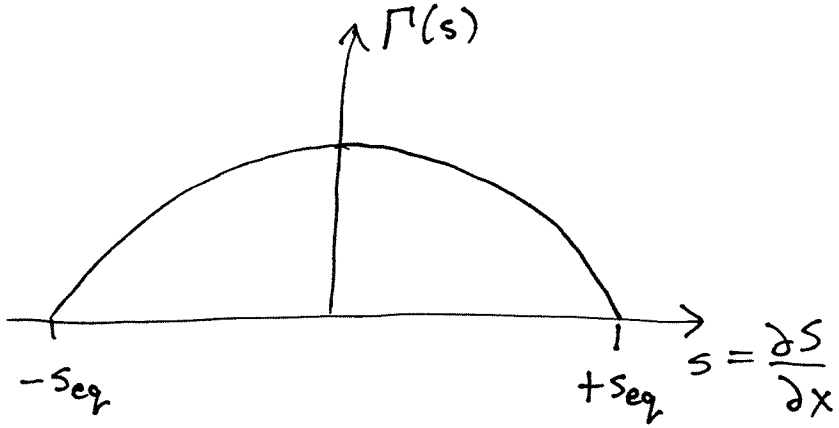
[More generally, this is also coupled to transport of energy, particles ....]

Applies to "typical" states, not all states (like 2nd Law).

Jonay, H, Nahum (to appear '18)

$$\frac{\partial s}{\partial t} = \tfrac{1}{2} s_{eq} \Gamma\left(\frac{\partial s}{\partial x}\right) + \cdots$$

2nd law: $\Gamma(s) \geq 0$



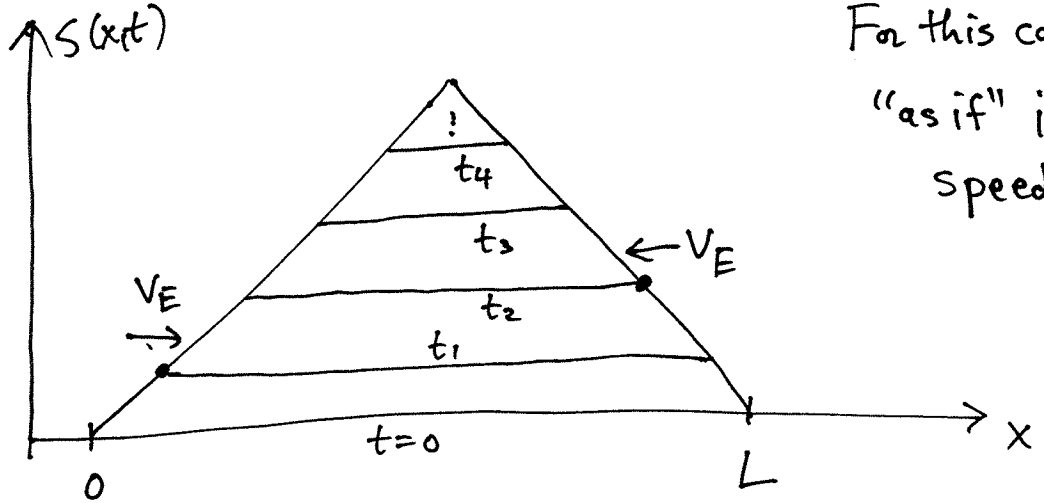
$\Gamma(s)$ is convex, $\Gamma(s_{eq}) = \Gamma(-s_{eq}) = 0$

$\Gamma(0) = v_E$ = "entanglement speed"

$s_{eq} \left.\frac{d\Gamma(s)}{ds}\right|_{-s_{eq}} = v_{LR} = v_B$ = "Lieb-Robinson" ('72) "butterfly" speed.

convexity: $v_E \leq v_{LR}$

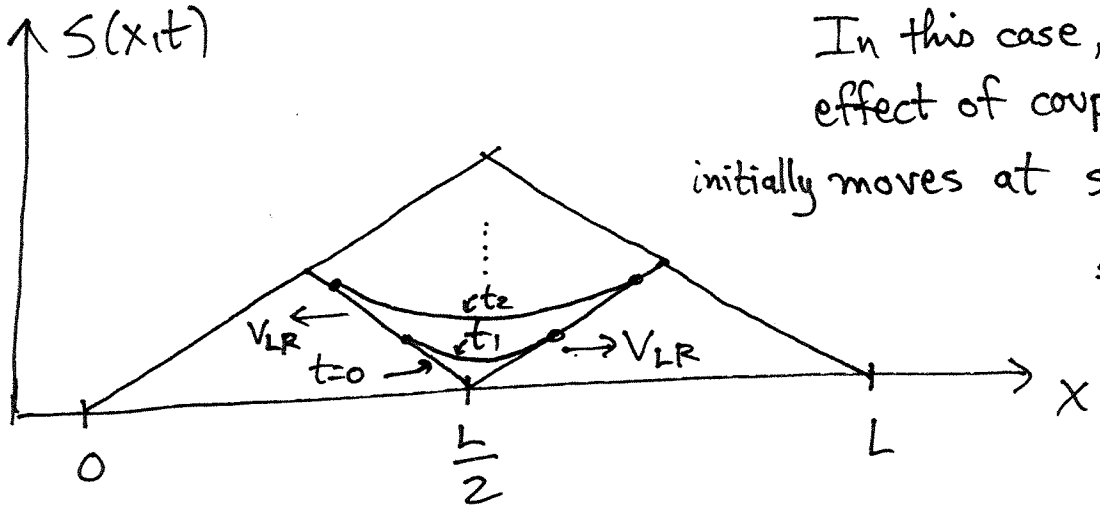
Example: start with a product state: $S(x, t=0) = 0$:



For this case, entanglement thermalizes "as if" it is "spreading" with speed v_E.

Kaufman, ..., Greiner exp'm't
Science 353, 794 (2016).
measured $S_2(x,t)$ for
6-atom, Bose-Hubbard chain.
6-site

Example: Start with a product of thermalized half-systems:



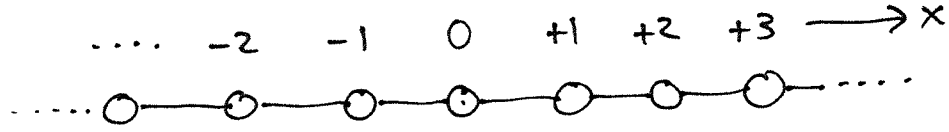
In this case, the entanglement effect of coupling the two half-systems initially moves at speed $V_{LR} = V_B$ = system's "speed limit".

These results are obtained analytically for random quantum circuits in $q \to \infty$ limit.

Numerics on more "realistic" models (e.g. Hamiltonian spin-1/2 chains) are consistent with these results.

Many-body chaos: First: Classical:

e.g. Chaotic classical spin chain (1711.07505) or anharmonic oscillator chain....



Time evolution can be: time-ind'p't H, or Floquet, or random circuit of classical "gates".

"Butterfly effect": Consider two initial states that differ infinitesimally only at site 0: S = "spin" $dS(0,0) \neq 0$ $dS(x,0=t) = 0$ for $x \neq 0$

Both states evolve by same $H(t)$. Measure resulting difference: $\frac{dS(x,t)}{dS(0,0)}$

Difference spreads and grows. But is of "random" sign due to chaos.

So square + average over initial states:

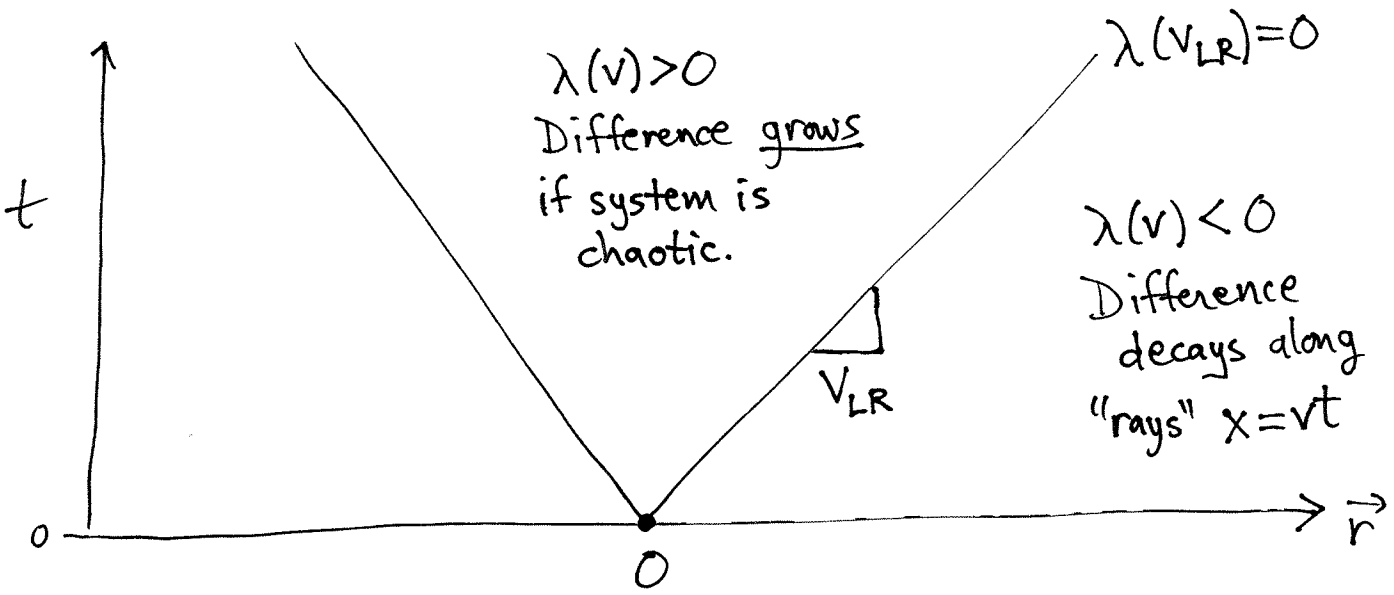
$$C(x,t) \equiv \left\langle \left(\frac{dS(x,t)}{dS(0,0)} \right)^2 \right\rangle \equiv \text{"O.T.O.C."}$$

"Out of Time Order Correlation"

Analog of quantum "out-of-time-order commutator":

$$\left\langle \left| [\hat{S}(x,t), \hat{S}(0,0)] \right|^2 \right\rangle$$

What happens?



For $x=vt$ $\quad C(x,t) = \left\langle \left(\frac{dS(x,t)}{dS(0,0)} \right)^2 \right\rangle \sim e^{t\lambda(v)}$

grows (or decays) exponentially with velocity-dependent Lyapunov exponent $\lambda(v)$

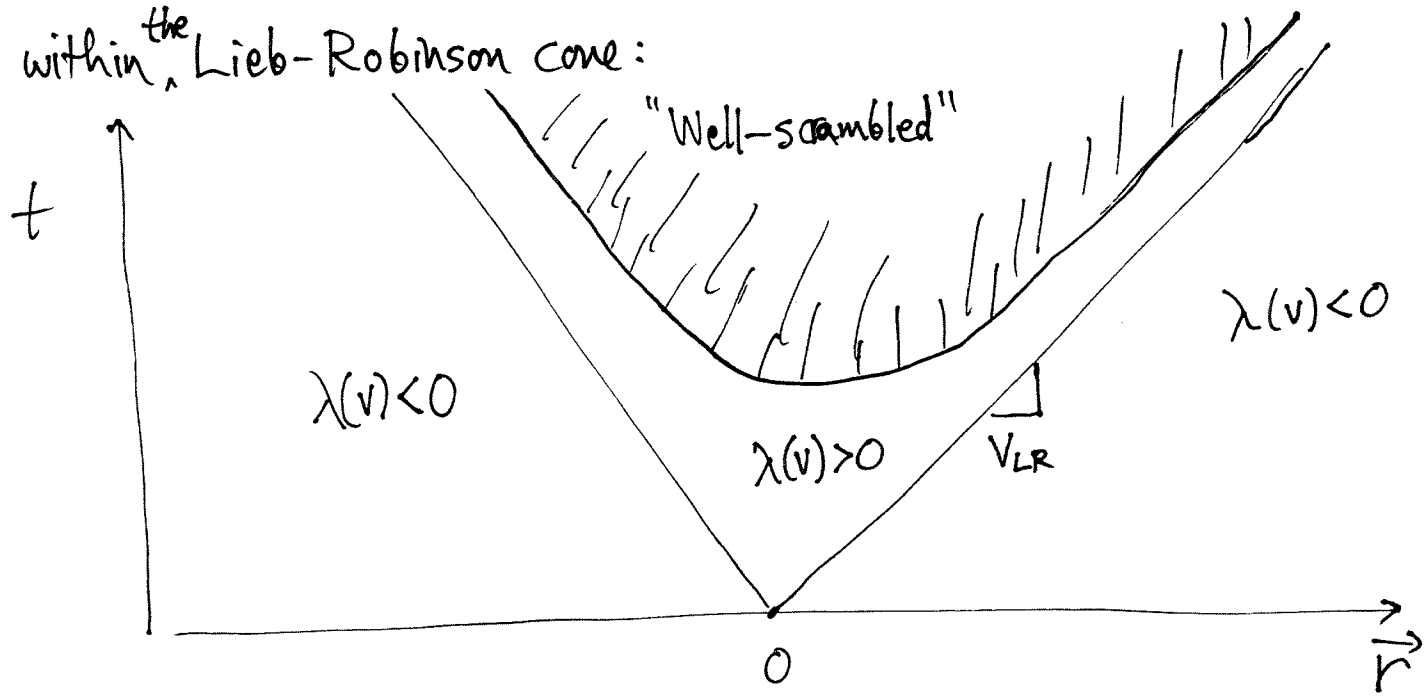
Lieb-Robinson '72
Kaneko '86
⋮

Chaos: $\lambda(v) > 0$ for $v < v_{LR} = v_B$ = Lieb-Robinson speed or "Butterfly speed".

Inside "Lieb-Robinson cone" $\lambda(v) > 0$: Difference grows exponentially.

v_{LR} set by $\lambda(v_{LR}) = 0$ (Can also have $\lambda(v) = 0$ for $v < v_{LR}$ for some nonchaotic integrable systems)

If we have a non-infinitesimal small initial difference $\delta S(0,0)$, then chaotic systems will "scramble" and "forget" their initial similarity within the Lieb-Robinson cone:



e.g., see 1711.07505 Das, et al.

"Fully quantum" system such as two states per site (spin-$\frac{1}{2}$):

can not make small initial local difference: $\lambda(v)>0$ regime does not exist.

($\lambda(v)>0$ regime does exist for certain large-N quantum models, such as SYK chains; Gu, Qi, Stanford 1609.07832).

Quantum: Operator spreading dynamics.

Start with initially local operator:

$A_{x=0}(t=0)$ that is the identity operator at all other sites, $\vec{r} \neq 0$

Trace $\{A_0\} = 0$ so $A_0(0)$ does not contain $\mathbb{1}_0$.

Time evolve with some given $H(t)$:

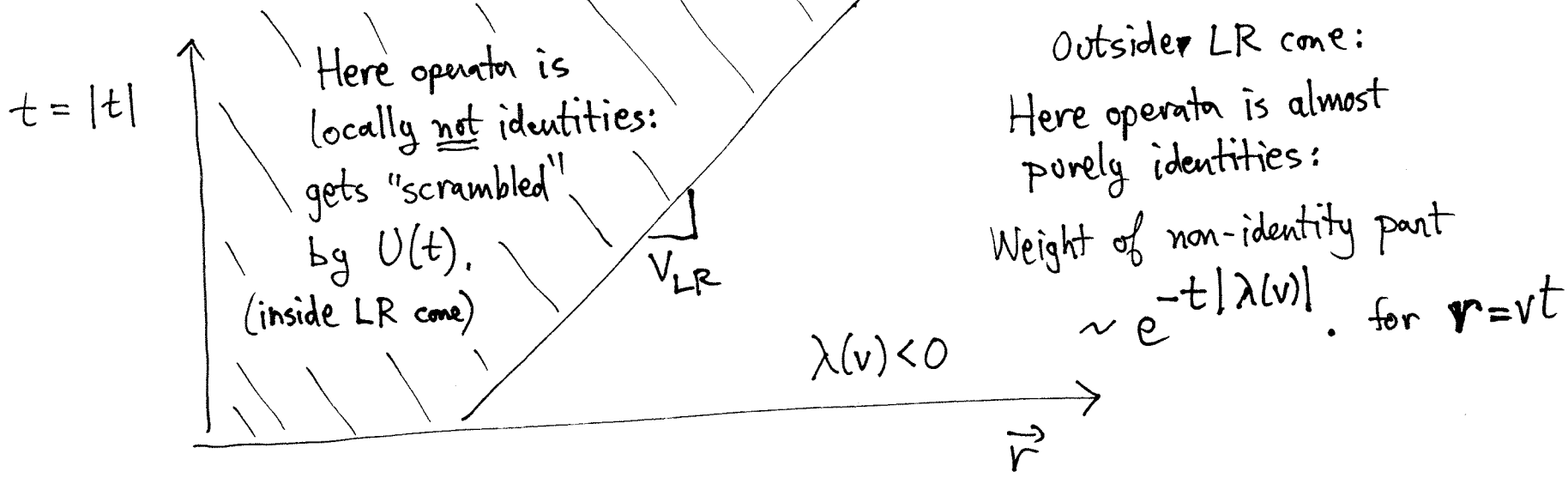
$$A_0(t) = U^\dagger(0,t)\, A_0(0)\, U(0,t)$$

In spatial regions where $A_0(t)$ is the local identity, nothing happens:

locally: $U^\dagger \mathbb{1} U = \mathbb{1}$

But operator "spreads" with increasing $|t|$:

"Edge" of spreading operator: $\lambda(v_{LR}) = 0$



$A_0(t) \cong$ (Nontrivial, highly-entangled "scrambled" operator within LR cone)

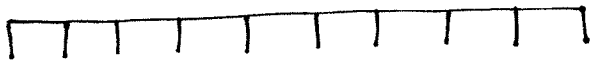
$\otimes$ (The identity operator outside of the LR cone)

More systematically: E.g., spin-1/2 chain

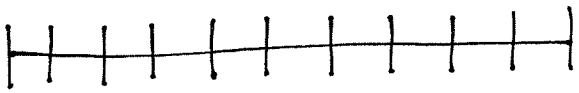
2 states per site ↑ or ↓

4 local Pauli operators per site: $\mathbb{1}, X, Y, Z$.

View operators as states on a doubled system:

State $|\psi\rangle = \sum_{n=1}^{2^L} c_n |n\rangle$: 

$n = \uparrow$ or $\downarrow$ on each site

Operator $\hat{B} = \sum_{n,m=1}^{2^L} |n\rangle B_{nm} \langle m|$: 

Basis of 4^L product operators ("Pauli strings"): $\hat{S} \overset{\text{e.g.}}{=} \cdots \otimes Y_0 \otimes \mathbb{1}_1 \otimes Z_2 \otimes \cdots$

Inner product: $\text{Trace}\left\{\frac{\hat{A}^\dagger \hat{B}}{2^L}\right\}$

States evolve as $|\psi(t)\rangle = U(0,t)|\psi(0)\rangle$: Contains nothing for U to cancel against: No locally "simple" states that have locally simple dynamics.

Operators: $A_0(t) = U^\dagger(0,t) A_0(0) U(0,t)$: No local dynamics if locally $[A, H] = 0$. Always true for local identity. Also true for conserved operators (if any).

Expand $A_0(t)$ in basis of Pauli strings:

$$A_0(t) = \sum_{\hat{S}} a_{\hat{S}}(t)\,\hat{S} \qquad \text{Trace}\left\{A_0^\dagger A_0\right\} = \text{constant} = 1$$

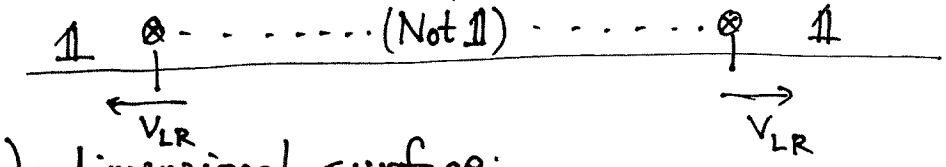
Unitarity: $\sum_{\hat{S}} |a_{\hat{S}}(t)|^2 = 1$ is a "conservation law"

Dynamics of the "edge" of the operator: Nahum, et al. 1608.06950, 1705.08975
v. Keyserlingk, et al. 1705.08910, 1710.09827

As operator spreads: weight $|a_{\hat{S}}(t)|^2$ moves to "longer" Pauli strings.

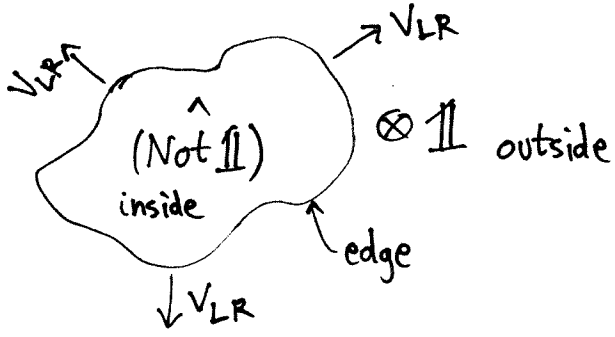
Each "string" $\hat{S}$ has an "edge", outside of which it is purely identities.

in 1D: left and right edges:



in $d>1$: edge of $\hat{S}$ is a $(d-1)$-dimensional surface:

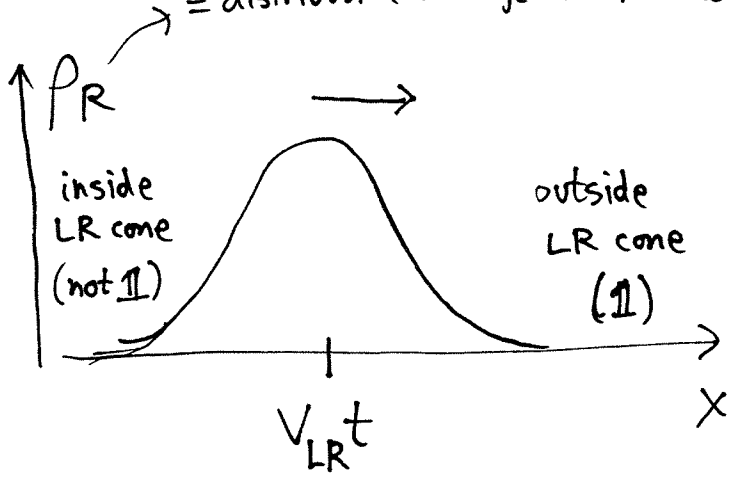
$\hat{S} =$



1D: look at density distribution of right edge:

$$\rho_R(x,t) = \sum_{\hat{S}} |a_{\hat{S}}(t)|^2$$

$\hat{S}$: right edge at x



$U^\dagger(dt)\,\hat{S}\,U(dt)$ has amplitudes for → making string shorter
→ leaving same length
→ making string longer

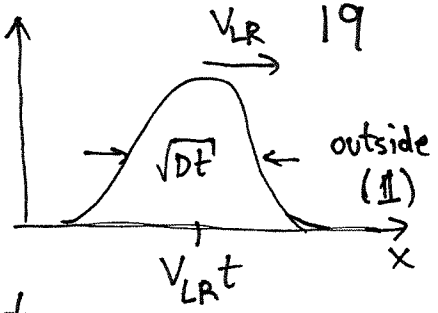
but biased to more longer than shorter: "entanglement happens".

edge distribution does a biased diffusion in 1D: drift speed $= V_{LR} = V_B$

$d>1$: Edge does nonlinear stochastic growth: KPZ: Kardar-Parisi-Zhang

captures coarse-grained edge dynamics.

In 1D: $\rho_R(x,t) \sim e^{-(x-V_{LR}t)^2/D_{edge}t}$



outside of LR cone: $x \geq V_{LR}t$ $\sim e^{t\lambda(v)}$ for $x = vt$

here ρ_R gives probability of non-identity

$\lambda(v) = -\frac{(v-V_{LR})^2}{D_{edge}} \leq 0$ for $v \geq V_{LR}$

= velocity dependent Lyapunov exponent

Large-N, large-q models, semi-classical limit: can have $D_{edge} \to 0$ (also higher-d) so edge moves more deterministically.

For "fully quantum" ($O(1)$-dimensional local Hilbert space)

$\lambda(v)$ well-defined at edge and outside of LR cone where $\lambda(v) \leq 0$.

Not yet clear how to define a positive Lyapunov exponent $\lambda > 0$ for "fully quantum" many-body chaos. What is it that grows exponentially by a large factor? (if anything?)

Emergence of dissipative dynamics

Khemani, Vishwanath, H.
1710.09835

How does irreversible dissipative dynamics (diffusion, viscosity ....) emerge from reversible unitary dynamics?

Diffusive example: $Z_{\vec{r}}$ is the "Pauli string" $\hat{S}$ that is Z at $\hat{r}$ and $\mathbb{1}$ at every other site.

Total Z: $M = \sum_{\vec{r}} Z_{\vec{r}}$ is conserved: $[U, M] = 0$

No other conservation laws: Z density diffuses (we have explicit quantum circuit models for this)

Unitarity: $\sum_{\text{all } \hat{S}} |a_{\hat{S}}(t)|^2$ is conserved.

Now also $\sum_{\vec{r}} a_{Z_{\vec{r}}}(t)$ is conserved. This constrains the spreading of the $Z_{\vec{r}}$ operators.

due to conservation of total Z.

Separate Pauli strings $\hat{S}$ into:

"conserved": the $\{Z_{\vec{r}}\}$, these are local observables and their spreading is constrained by total Z conservation

"non conserved": all other strings. These are mostly highly-nonlocal and thus in practice "hidden" (not observable).

Consider initial spin operator Z_0 at origin. at $t=0$: $a_{Z_0}^{(0)} = 1$ $a_{\hat{S}}(0) = 0$ for all other $\hat{S}$.

At later time this operator spreads to:

$$Z_0(t) = \underbrace{\sum_{\vec{r}} a_{Z_{\vec{r}}}(t) Z_{\vec{r}}}_{\text{conserved, observable}} + \underbrace{\sum_{\text{all other } \hat{S}}{}' a_{\hat{S}}(t) \hat{S}}_{\text{nonconserved mostly "hidden"}}$$

$$Z_0(t) = \sum_{\vec{r}} a_{Z_{\vec{r}}}(t) Z_{\vec{r}} + \sum_{\hat{S}}{}' a_{\hat{S}}(t) \hat{S}$$

Full dynamics of $Z_0(t)$ is unitary and reversible.

But, once a string $\hat{S}$ becomes highly nonlocal it becomes "hidden",
leaving a nonunitary dynamics of "observables".

Diffusion:

$$a_{Z_{\vec{r}}}(t) \sim \frac{1}{t^{d/2}} e^{-r^2/Dt}$$

(D: diffusivity of Z)

is "observable" part of operator remaining at late time.

Fraction of weight of $Z_0(t)$ that remains "observable":

$$\sum_{\vec{r}} |a_{Z_{\vec{r}}}(t)|^2 \sim \frac{1}{t^{d/2}} \to 0 \quad \text{at late time.}$$

The rest of its weight is in long "hidden" Pauli strings.

Dissipation is dynamic conversion of simple observable operators
to highly nonlocal, thus "hidden" operators.

SUMMARY of Lecture 1.

- Thermalization: Closed many-body system being a "thermal bath" for itself.
- Entanglement dynamics: local production of entanglement.
- Many-body chaos. The "Lieb-Robinson cone".
- Operator Spreading.
- Emergence of dissipation: the "hiding" of observables.